Euler Buckling

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Abstract:

In designing members in engineering it is of vital importance that the members can satisfy specific strengths, deflections and stability requirements (Hibbeler R.C., 2011). A column is a long, straight and slender member that is experiencing compressive axial loading. Catastrophic failure can result from the columns deflecting laterally at enormous levels. This effect is known as buckling. The purpose of this experiment is to investigate the effect of increasing load on members, with different supports, critical loading, the derivation of the load equations, buckling and effects of length of the members. Calculations of the moment of inertia (I) were necessary because of how irregular beams are in the real world an error of 0.12% for the shorter column and 2.76% for the longer column. Figure (6) showed a direct proportion relationship between the load and deflection for a beam resting horizontally from the graph the elastic modulus obtained was 4.3% from the actual value. Loading columns while vertical produced asymptotic graphs of the load against the deflection with the both graphs approaching critical loading value Figure (8). The graphs produced reliable results as would be expected from Euler buckling. Errors of 4.75% and 2.77% were recorded between the theoretical and experimental values of the 550mm and 750mm column respectively. Such errors are acceptable but it is important to note that in the case of the 750mm column the elastic modulus was assumed to be 200GPa as noted by the demonstrator. Concept of slenderness ratio reviewed and the effect of lengths given that all the other variables remain constant.
<table>
<thead>
<tr>
<th>Contents</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Contents</td>
<td></td>
</tr>
<tr>
<td>Introduction</td>
<td>3</td>
</tr>
<tr>
<td>Theory</td>
<td>4</td>
</tr>
<tr>
<td>Procedures and Equipment</td>
<td>6</td>
</tr>
<tr>
<td>Results</td>
<td>8</td>
</tr>
<tr>
<td>Discussion</td>
<td>10</td>
</tr>
<tr>
<td>Conclusion</td>
<td>11</td>
</tr>
<tr>
<td>References</td>
<td>12</td>
</tr>
<tr>
<td>Appendix</td>
<td>13</td>
</tr>
</tbody>
</table>
Introduction

A column is a structural member experiencing compressive loading at the either end, given however, that the cross sectional dimensions are considerably smaller than the length which will be the direction in which a load is applied. The cross-sectional area is to be kept constant in order to analyse the possible deflection. Buckling is the phenomenon that happens when a column is experiencing an axial load and deflects due to the loading being big enough. Buckling can lead to failure if the compressive loading is big enough. It is important to note that buckling failure is not as a result of the material since after applying the loads the material retains its original shape hence the elastic limits will not have been reached. So buckling failure occurs mainly to loads that are smaller than the elastic yield strengths. To engineers it is important to be able to predict buckling levels due to how destructive, dangerous and sudden it can occur.

The critical load of a column is the maximum axial load that a column can support before failure and any load greater than the critical load will cause the beam to deflect laterally and bow out.

Masses of the beams are neglected and focus is on the elastic modulus and the cross sectional area. Equation (1) shows the moment of a beam about a certain distance from the support, the beams in this experiment will be either pinned or fixed at both ends. An ideal column is assumed meaning that the column is made of homogeneous material, is perfectly engineered (cross-sectional area constant throughout the structure), material is linearly elastic and the load is applied exactly in the middle of the cross-sectional area. Upon reaching the critical load the column tends to be unstable and the structure becomes reliant on how well the column can restore its natural structure.

This report analyses the theoretical and experimental results of the out of plane deflection which results when the struts are sustaining a load axially. A close to ideal beam situation is also assumed meaning that, the beam is perfectly uniform and loaded also that the material in the beam is homogeneous. This however, in the real world cannot be the case because of lack of accurate precision from manufacturing equipment and uneven distribution of load around the beam.

When a column is exposed to compressive loading starting from zero, the column would start at a state of balance (equilibrium). Small lateral deflections occur so long as this state of balance is not exceeded. Meaning that the beam remains below it elastic limitations. Critical loading is the maximum compressive loading that a column can take before reaching unstable equilibrium. Any further increase in loading would result in catastrophic failure. Critical loading for long and slender columns occurs below the elastic limits.
Theory

The theory of buckling of columns under a compressive axial load was discovered by Leonard Euler (1707-1783). Columns are to remain long, straight and slender with specific supports at the ends. A compressive load $P$ is applied and has to pass through the centre of the cross-sectional area. If a column is pinned the pins are assumed frictionless. Ideal beams are assumed meaning that these columns have no defects. Figure (1) is of an ideal column which is symmetric with deflections occurring only in one plane.

![Figure 1(a) Buckling of a pin-jointed column under an axial load P, (b) the Free-body force diagram showing internal forces (Civil Engineering Lectures., 2013)](image)

If a pin jointed beam is experiencing a certain load $P$ while taking a distance $x$ from the top pin and a displacement $(y(x))$. The bending moment $M(x)$ about the point will be the load multiplied by the displacement $(P*y(x))$. Applying this to the equation for beams we would have an expression of modulus of elasticity and load times the deflection. The derivation of this equation is beyond this report.

Instead of using $y(x)$ to denote deflection $\delta$ will be used;

$$EI \frac{d^2 \delta}{dy^2} = -P\delta \tag{1}$$

Where $E$ is modulus of elasticity and $I$ is the area moment of inertia

Applying differential equation for the column would see the introduction of new term $k^2$, which is represented in Equation (1.1) the full derivation of the differential equation is found in the Appendix and the final solution would show the load equation as shown by Equation (2).

$$\frac{d^2 \delta}{dy^2} = -\frac{p}{EI}\delta = -k^2\delta \tag{1.1}$$

$$P = \frac{n^2\pi^2}{l^2}EI \tag{2}$$

Where $n=1, 2, 3, \ldots$ depending on the support.
The above analysis helps in determining the minimum load $P$ at which buckling occurs. In this experiment the beams are either pin-ended or fixed which means that $n=1$ or 2 respectively and substituting these $n$ values would give critical loads. This critical load can also be called the Euler buckling load.

$$P_C = \frac{\pi^2 E I}{L^2}$$ \hspace{1cm} (3)

$$P_C = \frac{4\pi^2 E I}{L^2}$$ \hspace{1cm} (4)

In the introduction it was noted that the critical load is independent from material strength. Equations (3) and (4) shows that the critical load is dependent on the modulus of elasticity $E$, moment of inertia $I$ and length $l$. The first buckling mode is related to $n=1$ and is given in Equation (3), so a different value of $n$ would mean that theoretically an increase, but this is not the case since columns buckle once they have reached the critical load so $n$ has no practical interest. Figure (2) shows different supports to 4 columns.

![Figure 2. Buckling on columns with different support systems (Birch D., 2012)](image)

If a column has a rectangular cross-section as shown in Figure (3) below would fail in a certain way.

![Figure 3. Rectangular cross-sectional area of a column (Den Hartog J.P. 1949. Pg 56)](image)

Buckling failure will occur at the lowest value of moment of inertia. Figure (3) would buckle about the $x$-plane rather than the $y$-plane. Achieving a balance that the ($l_x$ approx equal $l_y$) would result in better preferred columns. Calculation of moment of inertia is dependent on the width and height, both $I$’s should be calculated to obtain the smaller value.
Euler buckling load for ideal conditions is reached instantaneously and the failure is immediate, but this is not the case in the real world deflections are noted with the increase in load until reaching a critical load as shown by Figure (5).

Figure 5: A column under a load (a); ideal Euler load (P), deflection ($\delta$) curve (b); actual observed results (c)  (Henslee E. and Ward S., 2013)

When a beam is loaded mid-span calculations of bending moments would leave a relationship between the load $P$ and deflection. Equation (6) shows a direct proportion relation between the load and the deflection. Figure (6) in the Results section shows a straight line from the first experimental results.

$$P = \frac{4\pi^2 EI}{L^2} \delta$$  \hspace{1cm} (6)

The Euler buckling and the arguments presented earlier will only work if the material behaviour stays elastic. Moment of inertia ($I$) can be defined as the cross-sectional area $A$ and the minimum radius of gyration $r$.

$$I = Ar^2$$  \hspace{1cm} (7)

Substituting Equation (8) into Equation (3), then diving both sides by the area a formula for critical stress for a column $Pc/A$, taking also the minimum radius of gyration ($r$) as a safety feature since ($I$) was minimum. From the critical stress a ratio between the length and the radius of gyration ($L/r$) would determine the critical stress since the critical load, cross sectional area and elastic modulus are all constants. Equation (9) will be always applicable so long as the material limits are not exceeded.

$$\sigma_c = \frac{Pc}{A} = \frac{\pi^2 E}{(L/r)^2}$$  \hspace{1cm} (8)
The slenderness ratio is how flexible a column can be, this helps explain the actual effect of length (long, intermediate or short). Figure (6) shows the relation while the material remains below the proportional limits hence the column still behaves in an elastic manner. If the slenderness ratio from Figure (6) is less than 89 \((L/r)<89\) the yield stress will be exceeded before buckling so the Euler formula can not be used. Table (1) in the appendix also shows the lowest slender ratio for different types of steels.

Procedures and Equipment

**Equipment**
Steel beam 1 dimensions (550x19x3)mm
Steel beam2 dimensions (750x19x3)mm
Ruler
Micrometre Dile
Micrometre screw gauge
Strut rig machine
Masses

**Procedure**
Before starting the experiments, instruments have to be calibrated, with the smallest degree of measurement noted for error calculations. Three experiments were carried out in this lab and each case is explained in this section. The rig was adjusted to fit the columns for each experimental case Figure (7) shows the strut machine used in the experiment. Table (1) shows the modulus of elasticity and vital in error calculations.

Case 1: Horizontal lying knife edge (pin) setup (550mm)
The 550mm column was placed horizontally on the knife edge and the rear of the positions of the strut gauge adjusted. On the marked mid-span a stirrup was placed perpendicular to the length of the bar, here loads were going to be attached on the load carried suspended underneath the stirrup. The dial gauge was mounted vertically above the stirrup just touching and calibrated to zero carefully. Loads were carefully applied starting with 1N until 10N and the deflection given on the dial noted in Table (2). A plot was made of the load (W) against the deflection \((\delta)\). Since the moment of inertia has been calculated, length known the gradient
of this curve can be used to find the modulus of elasticity. The working out and further explanation is carried out in the results section. The elastic modulus can be obtained from calculating bending moments or graphically (gradient of the slope) Figure (6).

Case 2: Vertical lying knife edge (pin) setup (550mm column)
The piston was first adjusted to releases any forces. The bar was carefully taken and placed vertically on knife edge supports. The dial was placed horizontally and at the mid-span of the column. Pressure was applied to make sure the beam deflected away from the dial. The load was taken off completely and the dial reset to zero. The starting deflection readings were taken at 25N and recorded in Table (3). A graph was plotted of the load $W$ against the displacement $\delta$ and comparisons made against the Euler formula this is also carried out in the results section.

Case 3: Vertical lying clamped (fixed) setup (750mm)
The strut rig was adjusted to fit the 750mm span. Using the rule the middle of the span was determined and the column was fixed into position using the clamps. The piston was turned to allow load and observe if the column deflected away from the dial. The pressure was taken off after the observation and the dial was calibrated to zero. Since the length of the column had increased increments began at 50N and deflections were recorded in Table (3), and a curve was plotted of load $W$ against displacement $\delta$.

Results

A column is never ideal, so using a micrometre on three different points of the span dimensions of width and height can be taken and averaged as shown in Table 4 below. Equation (10) was used to calculate the means.

<table>
<thead>
<tr>
<th>Beam</th>
<th>Width (b) ±0.005mm</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>AVERAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>550mm</td>
<td></td>
<td>19.95</td>
<td>19.97</td>
<td>19.97</td>
<td>19.96</td>
</tr>
<tr>
<td></td>
<td>Height (d) ±0.005mm</td>
<td>2.95</td>
<td>2.99</td>
<td>2.92</td>
<td>2.95</td>
</tr>
<tr>
<td>750mm</td>
<td></td>
<td>19.92</td>
<td>19.92</td>
<td>19.92</td>
<td>19.92</td>
</tr>
<tr>
<td></td>
<td>Height (d) ±0.005mm</td>
<td>2.98</td>
<td>2.98</td>
<td>2.98</td>
<td>2.98</td>
</tr>
</tbody>
</table>

$$\mu = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} x_i$$

(10)

Using the aide of Table (4) and Equation (6.1), the moment of inertia was calculated and recorded.
The loads and deflections of the horizontal 550mm column on a knife edge support were carefully measured and recorded in Table (1). A plot of the load $W$ against the deflection $\delta$ showed a linear relationship with all the points lying on the straight line apart from one of the points. This agrees with Equation (6), which shows that the line should cross the x-axis at zero and the constants in the equation making up the gradient. The slope can be calculated as $\frac{\Delta \text{load}(W)}{\Delta \text{deflection}(\delta)}$ which is 0.21N/mm. Since all the other quantities making up the gradient are already known for the experimental values apart from the Elastic modulus we can use Equation (8) to find the experimental value of the elastic modulus.

$$E = \frac{L^3}{0.21 \times 48 \times I}$$

Substituting the values of length and moment of inertia using this value the Elastic modulus would be 209GPa. A 4.5% error is obtained between the theoretical and experimental value of the elastic modulus. Figure (7) shows the values from Case 1.

![Figure 7. Load against deflection with 5% error bars and a line of best fit](image)

In Case 2 the 550mm beam was pinned and load was gradually applied, results are tabulated in Table (3). Knowledge obtained from the Theory section that when a column is pinned Equation (3) can be used to find the critical load. Table (3) also contains the values of the fixed 750mm beam applying Equation (4) since $n=2$ would give the experimental values of the critical load.

Table 5: Experimental moments of inertia (I) including instrument error

<table>
<thead>
<tr>
<th>Column</th>
<th>550mm</th>
<th>750mm</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$42.70\pm(6.25\times10^{-10})$mm$^4$</td>
<td>$43.93\pm(6.25\times10^{-10})$mm$^4$</td>
</tr>
</tbody>
</table>

Table 5: Critical loading experimental values of 550 and 750mm columns

<table>
<thead>
<tr>
<th>Column</th>
<th>Pinned (n=1)</th>
<th>Fixed (n=4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>550mm</td>
<td>291.2N</td>
<td>-</td>
</tr>
<tr>
<td>750mm</td>
<td>-</td>
<td>616.64N</td>
</tr>
</tbody>
</table>
The values from Table (5) show the experimental values for both columns with their different supports. The theoretical values are calculated in the Appendix; errors of 4.75% and 2.77% were obtained between the 550mm and 750mm column respectively.

The results obtained from the two experiments shows graphs that are asymptotic after reaching a certain load. The 750mm column has a steeper slope and reaches the maximum loading of 600N. The load $W$ against deflection $\delta$ graphs for both the 550mm span and 750mm column are plotted in Figure (9) which is even more accurate in its approach to the critical values than the experimental calculation.

![Figure 10: Load against deflection comparison of (750 and 550)mm columns including 5% error propagation](image)

**Discussion**

Human errors are inescapable whenever an experiment is carried out. Tackling this however, would be best done by tasking individuals to certain tasks and repeating those tasks three or more times. Hysteresis error is visible on the strut machine which was slow to react to initial inputs of force. There is a 4.5% error between the theoretical modulus of elasticity and the experimental value, reduction of this error would be achieved by repeating the experiment three or more times also using a more sensitive dial. Errors of 0.12% and 2.76% were recorded for the moment of inertia of the 550mm and 750mm beams respectively. For the 550mm beam such an error is within reason but increasing the points where the readings are taken would certainly lower the error.

A 4.75% and 2.77% error was obtained for the critical loading of the 550mm and 750mm beam. Again such errors are acceptable, but however, it is important to note that for the 750mm column the elastic modulus was assumed to be 200GPa and not calculated.
From all the errors that are obtained the results are overall with reason and solidify that the experimental methods used are a good way in obtaining the critical loading. The small error of 4.75% for the 550mm column also reinforces the fact that different elastic modulus do not play much of an effect to the critical loading. The modulus that was obtained from Figure (7) was very good and coincides with those in Table (1). To add Figure (8) and (9) produced an even more accurate estimate of the critical load.

Figure (6), shows that buckling will occur if the material is long and remain with the elastic limits. Shorter columns have higher buckling ratios than longer, slender columns and the relationship between the critical stress and length\(^2\) is inverse which means there is an accelerated fall in the critical stress as the length increases assuming that the radius of gyration stays constant and the graph obtained is hyperbola. It is also worth noting that Euler buckling does not act as a safety factor but just the maximum load a column a beam can take before bowing out.

**Conclusion**

Upon starting this experiment the demonstrator had mentioned that the elastic modulus of both columns was 200GPa, and Case 1 gave a result which was very close to the actual value of the modulus. All the measurement errors from instrument calibrations were in cooperated in the calculations. Ideal conditions were assumed for the columns although this is not possible in reality, from these calculations however the small percentage errors between the theoretical and experimental values were still very reasonable.

Improvements can be definitely made to the experiment starting by using more modern instrument that have no hysteresis errors due to over use, or maybe even laser dials can be used to measure the deflection in that way the possibility of the dial falling off will be eliminated.
References

AULD D.J. (2010). Buckling of columns. Available:

Civil Engineering Lectures. (2013). Buckling of columns. Available:


Birch D., 2009, Design and Component production; A reference guide for engineering students, Surrey University

Engineering toolbox. (2010). Young modulus. Available:
Appendix

Important tables:

**Table 1.** Limitations of Euler’s formula (Jensen A and Chenoweth H.H, 1983. pg327)

<table>
<thead>
<tr>
<th>Material</th>
<th>Modulus of Elasticity, (MPa)</th>
<th>Proportional limit (MPa)</th>
<th>Lowest value of L/k for which Euler’s Formula Applies</th>
</tr>
</thead>
<tbody>
<tr>
<td>Structural steel, low alloy</td>
<td>200,000</td>
<td>342</td>
<td>76.0</td>
</tr>
<tr>
<td>Structural steel, carbon</td>
<td>200,000</td>
<td>214</td>
<td>96.0</td>
</tr>
<tr>
<td>Douglas fir, select structural</td>
<td>11,000</td>
<td>31.0</td>
<td>59.2</td>
</tr>
</tbody>
</table>

After carrying out the experiment results were collected and recorded in tables.

**Table 2.** 550x19x3mm steel column on horizontal knife edges (CASE I)

<table>
<thead>
<tr>
<th>Load ±0.5N</th>
<th>Deflection(δ) ±0.05mm</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.2</td>
</tr>
<tr>
<td>2</td>
<td>0.4</td>
</tr>
<tr>
<td>3</td>
<td>0.6</td>
</tr>
<tr>
<td>4</td>
<td>0.8</td>
</tr>
<tr>
<td>5</td>
<td>1.1</td>
</tr>
<tr>
<td>6</td>
<td>1.3</td>
</tr>
<tr>
<td>7</td>
<td>1.5</td>
</tr>
<tr>
<td>8</td>
<td>1.7</td>
</tr>
<tr>
<td>9</td>
<td>1.9</td>
</tr>
<tr>
<td>10</td>
<td>2.1</td>
</tr>
</tbody>
</table>

**Table 3.** Load and deflection readings of a 550mm pinned column and 750mm fixed column

<table>
<thead>
<tr>
<th>550mm column pinned</th>
<th>Load (W) ±0.5N</th>
<th>Deflection(δ) ±0.05mm</th>
</tr>
</thead>
<tbody>
<tr>
<td>25</td>
<td>0.1</td>
<td>0.2</td>
</tr>
<tr>
<td>50</td>
<td>0.2</td>
<td>0.2</td>
</tr>
<tr>
<td>100</td>
<td>0.4</td>
<td>0.2</td>
</tr>
<tr>
<td>150</td>
<td>0.7</td>
<td>0.4</td>
</tr>
<tr>
<td>200</td>
<td>1.4</td>
<td>0.5</td>
</tr>
<tr>
<td>225</td>
<td>1.7</td>
<td>0.7</td>
</tr>
<tr>
<td>250</td>
<td>2.2</td>
<td>0.9</td>
</tr>
<tr>
<td>400</td>
<td>4.2</td>
<td>1.3</td>
</tr>
<tr>
<td>450</td>
<td>7.6</td>
<td>1.9</td>
</tr>
<tr>
<td>500</td>
<td>13.9</td>
<td>2.9</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>750mm column fixed</th>
<th>Load (W) ±0.5N</th>
<th>Deflection(δ) ±0.05mm</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>0.2</td>
<td>0.2</td>
</tr>
<tr>
<td>100</td>
<td>0.4</td>
<td>0.2</td>
</tr>
<tr>
<td>150</td>
<td>0.7</td>
<td>0.4</td>
</tr>
<tr>
<td>200</td>
<td>1.4</td>
<td>0.5</td>
</tr>
<tr>
<td>225</td>
<td>1.7</td>
<td>0.7</td>
</tr>
<tr>
<td>250</td>
<td>2.2</td>
<td>0.9</td>
</tr>
<tr>
<td>400</td>
<td>4.2</td>
<td>1.3</td>
</tr>
<tr>
<td>450</td>
<td>7.6</td>
<td>1.9</td>
</tr>
<tr>
<td>500</td>
<td>13.9</td>
<td>2.9</td>
</tr>
</tbody>
</table>
Following the experiment Case 2 and 3, Figure (8) and (9) were plotted for the load against deflection of the 550mm and 750mm columns

Figure 6. Strut rig machine and a column fixed vertically

Figure 8. Plot of load against deflection of 550mm column vertically pinned
**Important Equations in the paper:**

\[ EI \frac{d^2 \delta}{dy^2} = -P\delta \]  \hspace{1cm} (A1)

Where E is modulus of elasticity and I is the area moment of inertia

Introducing partial differential equation formula and a constant \( k \);

\[ \frac{d^2 \delta}{dy^2} = -\frac{P}{EI} \delta = -k^2 \delta \]  \hspace{1cm} (A1.1)

\[ k = \frac{P}{EI} \]  \hspace{1cm} (A1.1a)

Removing the \(-k^2 \delta\) to the left hand side and adding it to the second partial differential

\[ \frac{d^2 \delta}{dy^2} + k^2 \delta = 0 \]  \hspace{1cm} (A1.1b)

Applying the knowledge of second order partial differential equations, the general solution to the equation

\[ \delta = A\cos(ky) + B\sin(ky) \]  \hspace{1cm} (A1.2)

Where A and B are constants which can be determined from the boundary conditions which are

\[ \delta(0) = 0 \text{ and } \delta(L) = L \]

Hence

\( A = 0 \) and \( 0 = B\sin(kL) \)

If \( B = 0 \), this would mean that the beam is not being deflected. But for any other value of B, would mean that \( \sin(kL) = 0 \), would mean that \( kL = n\pi \), where the n is an integer. Rearranging this would lead us having a definition of solution for k.

\[ k = \frac{n\pi}{L} \]  \hspace{1cm} (A1.2a)

Substituting back into Equation (1.1) and rearranging
\[ P = \frac{n^2\pi^2}{L^2}EI \]  
(A2)

The Euler load for a column which is pinned at both ends has \( n=1 \). Hence the formula for column with both ends pinned is

\[ Pc = \frac{\pi^2}{L^2}EI \]  
(A3)

For a column that is fixed at both ends \( n=2 \). So the formula for a column with both ends fixed

\[ Pc = \frac{4\pi^2}{L^2}EI \]  
(A4)

Moment of inertia, were the

\[ I_x = \frac{db^3}{12} \]  
(A5.1)

\[ I_y = \frac{bd^3}{12} \]  
(A5.2)

\[ P = \frac{48EI}{L^3}\delta \]  
(A6)

Moment of inertia by definition is:

\[ I = Ar^2 \]  
(A7)

Substituting Equation (7) into Equation (3)

\[ Pc = \frac{\pi^2EAr^2}{L^2} = \frac{\pi^2EA}{(L/r)^2} \]  
(A7.1)

Dividing the critical load by the area, critical stress

\[ \sigma_c = \frac{Pc}{A} = \frac{\pi^2E}{(L/r)^2} \]  
(A8)

To calculate mean

\[ \mu = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} x_i \]  
(A9)

Theoretical values of moment inertia (I):

550mm ideal beam: \( I = \frac{db^3}{12} = \frac{(19\times3^3)mm^4}{12} = 42.75mm^4 \)

750mm ideal beam: \( I = \frac{db^3}{12} = \frac{(19\times3^3)mm^4}{12} = 42.75mm^4 \)

Theoretical values of critical load (Pc) if E=200GPa

550mm pinned beam: \( Pc = \frac{\pi^2EI}{L^2} = \frac{\pi^2\times200\times10^3\times42.75}{550^2} = 278N \)
750mm fixed beam: \( P_c = \frac{4\pi^2 E I}{L^2} = \frac{4\times\pi^2 \times 200 \times 10^3 \times 42.75}{750^2} = 600N \)

Calculating Errors:

Elastic modulus for 550mm: \( Error = \frac{100\times(\text{theoretical} - \text{experimental})}{\text{theoretical}} = \frac{100(200 - 209)}{200} = 4.5\% \)

550mm moment of inertia: \( Error = \frac{100\times(\text{theoretical} - \text{experimental})}{\text{theoretical}} = \frac{100(42.75 - 42.7)}{42.75} = 0.12\% \)

750mm inertia (I): \( Error = \frac{100\times(\text{theoretical} - \text{experimental})}{\text{theoretical}} = \frac{100(42.75 - 43.93)}{42.75} = 2.76\% \)

Critical load on 550mm column pinned: \( Error = \frac{100\times(278 - 291.2)}{278} = 4.75\% \)

Critical load on 750mm column fixed: \( Error = \frac{100\times(600 - 616.64)}{600} = 2.77\% \)

Standard deviation error: \( s = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(\bar{x} - x_i)^2} \)

Standard deviation error of 550mm and 750mm columns:

<table>
<thead>
<tr>
<th></th>
<th>550mm</th>
<th>750mm</th>
</tr>
</thead>
<tbody>
<tr>
<td>Width (d)</td>
<td>0.01224</td>
<td>0</td>
</tr>
<tr>
<td>Height (b)</td>
<td>0.03536</td>
<td>0</td>
</tr>
</tbody>
</table>